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Qualitative properties of certain non-linear differential systems of second order

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Abstract

In this paper, we study the boundedness and square integrability of solutions in certain non-linear systems of differential equations of second order. We establish two new theorems, which include suitable sufficient conditions guaranteeing the boundedness and square integrability of solutions to the considered systems. The presented proofs simplify previous works since the Gronwall inequality is avoided which is the usual case. The technique of proof involves the integral test, and two examples are included to illustrate the results.

Keywords: Boundedness; Square integrability; Integral test; Non-linear; Differential system; Second order

1. Introduction

The study of qualitative behaviors of solutions, asymptotic behavior, stability, instability, boundedness, convergence, square integrability, etc., to differential equations of second order seems to be an important problem of the qualitative differential equations theory and has both theoretical and practical values in the literature. Numerous works were done on the subject (for example, see, the books or the papers of Ahmad and Rama Mohana Rao [1], El-Sheikh et al. [2], Gallot et al. [3], Grigoryan [4], Gu and Yu [5], Korkmaz and Tunc [6], Kroopnick [7], Pettini and Valdettaro [8], Kulcsar [11], Ogundare et al. [12], Sanchez [13], Shi [14], Tunc [15–18], Tunc and Tunc [19,20], Zhao [21] and the cited papers or books therein). However, we would not like to give here the details of the works and applications done regarding the mentioned qualitative properties.

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More recently, Napoles Valdes [9] considered the following second order scalar non-linear differential equation of the form

$$(p(t)x')' + f(t, x, x')x' + a(t)g(x) = q(t, x, x'). \tag{1}$$

Napoles Valdes [10] proved two new results concerning the boundedness and square integrability of the solutions of Eq. (1), under suitable assumptions. By this work, the author extended and improved the current literature through the sound mathematical analysis.

In [10], Napoles Valdes proved the following two theorems, respectively.

Theorem A ((Napoles Valdes [10,Theorem 1])). Assume that the following conditions hold:

(i) p and a are continuous functions on $I=[0, \infty)$ such that $0<p_0 \le p(t)<+\infty$ and $a_0 \le a(t) \le a_1 <+\infty$.
(ii) f is a continuous function on $I \times \Re^2$ satisfying $0<f_0 \le f(t, x, x')$.
(iii) g is also a continuous function for all x such that $xg(x)>0$ for $x \neq 0$ and $\int_0^{+\infty} g(x)dx = \infty$.
(iv) $|q(t, x, x')| \le e(t)$, where $e(t)$ is a non-negative and continuous function of t and satisfying $\int_0^{\infty} e(t)dt \le M < +\infty$, M is a constant. Then any solution $x(t)$ of Eq. (1), as well as its derivative, is bounded as $t \to \infty$ and $\int_0^{\infty} x'^2(t)dt < +\infty$.

Theorem B ((Napoles Valdes [10,Theorem 2])). Under hypotheses of Theorem A, we suppose that $xg(x)>g_0x^2$ for some positive constant $g_0>0$, and $0<p<p(t)<P<+\infty$, then all solutions of Eq. (1) are L^2 – solutions.

Napoles Valdes [10] proved both of Theorem A and Theorem B by the integral test.

In this paper, we take into consideration the following non-linear vector differential equations of the second order:

$$(q(t)X')' + H(t, X, X')X' + a(t)X = Q(t, X, X') \tag{2}$$

and

$$(q(t)X')' + \Phi(t, X, X') + a(t)G(X) = Q(t, X, X'), \tag{3}$$

respectively, where $X \in \Re^n$, $t \in \Re^+$, $\Re^+=[0, \infty)$; $H(.)$ is an $n \times n$ – symmetric and real valued continuous matrix, $q(.): \Re^+ \to \Re$, with $q'(t)$ exits and is continuous, $a(.): \Re^+ \to \Re$, $\Phi(.): \Re^+ \times \Re^n \times \Re^n \to \Re^n$, and $Q(.): \Re^+ \times \Re^n \times \Re^n \to \Re^n$ are continuous functions.

The purpose of this paper is to improve and extend the obtained results of [10] for Eq. (1) to Eqs. (2) and (3). In addition, we give the proofs related to the boundedness and square integarble solutions and their derivatives, which are less complex and quite general than those in the literature, by the integral test. In addition, two examples are presented to illustrate and verify the applicability of the obtained results. This paper has a new contribution to the topic in the literature. This fact shows the novelty and originality of this paper. The results to be established here may be useful for researchers working on the qualitative theory of solutions.

2. Boundedness

The following lemma plays a key role in proving our main results.

Lemma ((Horn and Johnson [9])). Let A be a real symmetric $n \times n$ – matrix. Then, for any $X \in \Re^n$,

$$a_1||X||^2 \ge \langle AX, X\rangle \ge a_0||X||^2,$$

where a_0 and a_1 are the least and greatest eigenvalues, respectively, of A.

The first main problem of this paper is the following theorem.

Theorem 1. Given Eq. (2). We assume that there exist positive constants a_0, a_1, h_0 and q_0 such that the following assumptions hold:

(i) $a_0 \leq a(t) \leq a_1 <+\infty$, $0< q_0 \leq q(t) <+\infty$ for all $t \in \Re^+$,
(ii) $\lambda_i(H(t, X, X')) \geq h_0$ for all $t \in \Re^+, X, X' \in \Re^n$,
(iii) $\left\| Q(t, X, X') \right\| \leq |r(t)|$, $|r(t)| \in L^1[0, \infty)$ for all $t \in \Re^+$.

Then, any solution $X(t)$ of Eq. (2), as well as its derivative $X'(t)$ is bounded as $t \to \infty$ and $\int_0^\infty ||X'(t)||^2 dt < +\infty$.

Proof. It is clear from Eq. (2) that

$$q(t)X'' + q'(t)X' + H(t, X, X')X' + a(t)X = Q(t, X, X'), \tag{4}$$

Multiplying Eq. (4) by $X'(t)$ and then integrating the obtained estimate from 0 to t, it follows that

$$\int_0^t \left\langle q(s)X''(s), X'(s)\right\rangle ds + \int_0^t \left\langle q'(s)X'(s), X'(s)\right\rangle ds, + \int_0^t \left\langle H(s, X(s), X'(s))X'(s), X'(s)\right\rangle ds$$
$$+ \int_0^t \left\langle a(s)X(s), X'(s)\right\rangle ds = \int_0^t \left\langle Q(s, X(s), X'(s)), X'(s)\right\rangle ds. \tag{5}$$

Applying the integration by part to the first term into (5), we obtain

$$\int_0^t \left\langle q(s)X''(s), X'(s)\right\rangle ds = \frac{1}{2}q(s)\left\|X'(s)\right\|^2 | - \int_0^t \left\langle q'(s)X'(s), X'(s)\right\rangle ds$$
$$= \frac{1}{2}q(t)\left\|X'(t)\right\|^2 - \frac{1}{2}q(0)\left\|X'(0)\right\|^2 - \int_0^t \left\langle q'(s)X'(s), X'(s)\right\rangle ds.$$

In view of assumption (ii), we have

$$\int_0^t \left\langle H(s, X(s), X'(s))X'(s), X'(s)\right\rangle ds \geq h_0 \int_0^t \left\|X'(s)\right\|^2 ds.$$

When we apply the mean value theorem for integrals, we obtain

$$\int_0^t \left\langle a(s)X(s), X'(s)\right\rangle ds = a(t^*) \int_0^t \left\langle X(s), X'(s)\right\rangle ds = \frac{1}{2}a(t^*)\|X(t)\|^2 - \frac{1}{2}a(t^*)\|X(0)\|^2,$$

where $0 < t^* < t$.

By assumption (iii), it follows that

$$\int_0^t \left\langle Q(s, X(s), X'(s)), X'(s)\right\rangle ds \leq \int_0^t |r(s)| \left\|X'(s)\right\| ds.$$

By the Cauchy–Schwarz inequality for integrals, it can also be checked that

$$\int_0^t \left\langle Q(s, X(s), X'(s)), X'(s)\right\rangle ds \leq \int_0^t |r(s)| \left\|X'(s)\right\| ds \leq \left(\int_0^t r^2(s)ds\right)^{1/2} \left(\int_0^t \left\|X'(s)\right\|^2 ds\right)^{1/2}. \tag{6}$$

In view of the above discussion and (5), we can conclude that

$$\frac{1}{2}q(t)\left\|X'(t)\right\|^2 - \frac{1}{2}q(0)\left\|X'(0)\right\|^2 + h_0 \int_0^t \left\|X'(s)\right\|^2 ds + \frac{1}{2}a(t^*)\|X(t)\|^2 - \frac{1}{2}a(t^*)\|X(0)\|^2$$
$$\leq \left(\int_0^t r^2(s)ds\right)^{1/2} \left(\int_0^t \left\|X'(s)\right\|^2 ds\right)^{1/2}.$$

Hence, it is clear that

$$\frac{1}{2}q(t)\left\|X'(t)\right\|^2 + h_0\int_0^t \left\|X'(s)\right\|^2 ds + a(t^*)\|X(t)\|^2 \le \frac{1}{2}q(0)\left\|X'(0)\right\|^2 + a(t^*)\|X(0)\|^2$$
$$+\left(\int_0^t r^2(s)ds\right)^{1/2}\left(\int_0^t \left\|X'(s)\right\|^2 ds\right)^{1/2}. \tag{7}$$

Let $m(t) = \left(\int_0^t \left\|X'(s)\right\|^2 ds\right)^{1/2}$. When we divide both sides of (7) by $m(t)$, it follows that

$$m^{-1}(t)\left[\frac{1}{2}q(t)\left\|X'(t)\right\|^2 + h_0\int_0^t \left\|X'(s)\right\|^2 ds + a(t^*)\|X(t)\|^2\right]$$
$$\le m^{-1}(t)\left[\frac{1}{2}q(0)\left\|X'(0)\right\|^2 + a(t^*)\|X(0)\|^2\right] + \left(\int_0^t r^2(s)ds\right)^{1/2}. \tag{8}$$

It is now clear that the left hand side of (8) is positive. Therefore, if $\|X(t)\|$ increases without bound and the term

$$m^{-1}(t)\left[h_0\int_0^t \left\|X'(s)\right\|^2 ds\right] = h_0\left[\int_0^t \left\|X'(s)\right\|^2 ds\right]^{1/2}$$

is bounded by the right hand side of (7), and we can see that $\left\|X'(t)\right\|$ is square integrable and bounded. Besides, it follows form (8) that $\|X(t)\|$ must be bounded. Otherwise, the left hand side of (7) becomes infinity, which is a contradiction. Eventually, it can be easily shown that the solution can be extended to all for t in $\Re^+$. This case completes the poof of Theorem 1.□

Example 1. Let $n = 2$. We consider a special case of Eq. (2) as follows:

$$\left[\left(4+\frac{1}{1+t^2}\right)\begin{bmatrix} x_1' \\ x_2' \end{bmatrix}\right]' + \begin{bmatrix} 10+t+x_1^2+x_2^2 & 1 \\ 1 & 10+t+x_1^2+x_2^2 \end{bmatrix}\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} + (2+\exp(-t))\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$= \begin{bmatrix} (1+t^2+x_1^2+x_2^2)^{-1} \\ \cos t(1+t^2+x_1^2+x_2^2)^{-1} \end{bmatrix}, \quad t \ge 0.$$

It is clear that

$$q(t) = 4+\frac{1}{1+t^2}, \quad 0 < 3 = q_0 \le 4+\frac{1}{1+t^2} \le 5 < +\infty,$$

$$a(t) = 2+\exp(-t), \quad 2 \le a_0 \le a(t) \le 3 = a_1 < +\infty,$$

$$H(.) = \begin{bmatrix} 10+t+x_1^2+x_2^2 & 1 \\ 1 & 10+t+x_1^2+x_2^2 \end{bmatrix},$$

$$\lambda_1 = 9+t+x_1^2+x_2^2 \ge 9, \quad \lambda_2 = 11+t+x_1^2+x_2^2 \ge 11,$$

$$\lambda_i(H(t, X, X')) \ge 9 = h_0, \quad t \ge 0,$$

$$Q(.) = \begin{bmatrix} \dfrac{1}{1+t^2+x_1^2+x_2^2} \\ \dfrac{\cos t}{1+t^2+x_1^2+x_2^2} \end{bmatrix},$$

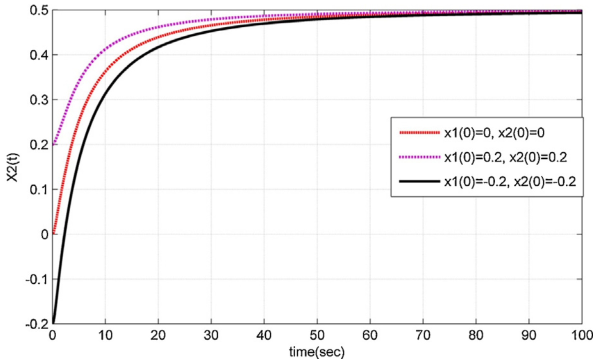

Fig. 1. Trajectory of $x_2(t)$ for Example 1.

$$\|Q(.)\| = \left\| \begin{bmatrix} \dfrac{1}{1+t^2+x_1^2+x_2^2} \\ \dfrac{\cos\, t}{1+t^2+x_1^2++x_2^2} \end{bmatrix} \right\| \le \frac{2}{1+t^2} = r(t),$$

$$\int_0^\infty |r(t)|\, dt = \int_0^\infty \frac{2}{1+t^2} dt = \pi,$$

that is, $|r(t)| \in L^1(0, \infty)$. Hence, all the conditions of Theorem 1 hold. Therefore, any solution $X(t)$ of the considered equation, as well as its derivative $X'(t)$ is bounded as $t \to \infty$ and $\int_0^\infty \left\|X'(t)\right\|^2 dt < +\infty$.

(Figs. 1 and 2).

Theorem 2. Given Eq. (3). We assume that there exist positive constants a_0, a_1, ϕ_0, q_0 and q_1 such that the following assumptions hold:

(i) $a_0 \le a(t) \le a_1 <+\infty$, $0< q_0 \le q(t) < q_1 <+\infty$ for all $t \in \Re^+$,
(ii) $\langle G(X), X\rangle \ge g_0\|X\|^2$, $\left\langle \Phi(t, X, X'), X)\right\rangle \ge \phi_0\|X\|^2$ for all $t \in \Re^+$, $X, X' \in \Re^n$,
(iii) $\left\|Q(t, X, X')\right\| \le |r(t)|$, $|r(t)| \in L^1[0, \infty)$ $t \in \Re^+$, $X, X' \in \Re^n$.

If $\left\|X'(t)\right\|$ is square integrable, then all solutions of Eq. (3) are L^2 – solutions, that is, $\int_0^\infty \|X(t)\|^2 dt < +\infty$.

Proof. In the light of the assumptions of Theorem 2, by following a similar way as shown in the proof of Theorem 1, we can conclude that $X'(t)$ is square integrable, that is, $\int_0^\infty \left\|X'(t)\right\|^2 dt < +\infty$.

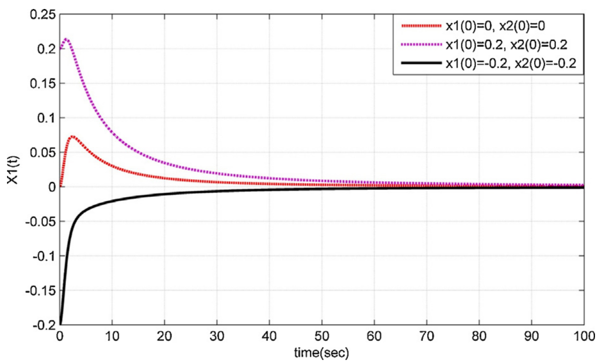

Fig. 2. Trajectory of $x_1(t)$ for Example 1.

Besides, it follows from Eq. (3) that

$$q(t)X'' + q'(t)X' + \Phi(t, X, X') + a(t)G(X) = Q(t, X, X'). \tag{9}$$

Multiplying Eq. (9) by $X(t)$ and then integrating the obtained estimate from 0 to t, we obtain

$$\begin{aligned}&\int_0^t \langle q(s)X''(s), X(s)\rangle\, ds + \int_0^t \langle q'(s)X'(s), X(s)\rangle\, ds + \int_0^t \langle \Phi(s, X(s), X'(s)), X(s)\rangle\, ds\\ &\quad + \int_0^t \langle a(s)G(X(s)), X(s)\rangle\, ds = \int_{\sim 0}^t \langle Q(s, X(s), X'(s)), X(s)\rangle\, ds.\end{aligned} \tag{10}$$

By noting the first and second terms in (10) and applying the integration by parts to the first term, we have

$$\begin{aligned}&\int_0^t \langle q(s)X''(s), X(s)\rangle\, ds + \int_0^t \langle q'(s)X'(s), X(s)\rangle\, ds = \langle q(t)X'(t), X(t)\rangle - \langle q(0)X'(0), X(0)\rangle\\ &\qquad - \int_0^t \langle q(s)X'(s), X'(s)\rangle\, ds - \int_0^t \langle q'(s)X'(s), X(s)\rangle\, ds + \int_0^t \langle q'(s)X'(s), X(s)\rangle\, ds\\ &\quad = \langle q(t)X'(t), X(t)\rangle - \langle q(0)X'(0), X(0)\rangle - \int_0^t \langle q(s)X'(s), X'(s)\rangle\, ds.\end{aligned}$$

It is also clear that

$$-\int_0^t \langle q(s)X'(s), X'(s)\rangle\, ds \geq -q_1 \int_0^t \|X'(s)\|^2 ds.$$

If we apply the mean value theorem for integrals and use the assumption $\langle G(X), X\rangle \geq g_0\|X\|^2$, we obtain

$$\int_0^t \langle a(s)G(X(s)), X(s)\rangle\, ds = a(t*) \int_0^t \langle G(X(s)), X(s)\rangle\, ds \geq a_0 g_0 \int_0^t \|X(s)\|^2 ds,$$

where $0 < t* < t$.

By the assumption $\langle \Phi(t, X, X'), X)\rangle \geq \phi_0\|X\|^2$, it is clear that

$$\int_0^t \langle \Phi(s, X(s), X'(s)), X(s)\rangle\, ds \geq \phi_0 \int_0^t \|X(s)\|^2 ds.$$

It is also obvious that

$$\int_0^t \langle Q(s, X(s), X'(s)), X(s)\rangle\, ds \leq \int_0^t |r(s)|\ \|X(s)\|\, ds.$$

The combination of the obtained estimates into (10) implies that

$$\begin{aligned}&\langle q(t)X'(t), X(t)\rangle - \langle q(0)X'(0), X(0)\rangle - q_1 \int_0^t \|X'(s)\|^2 ds + \phi_0 \int_0^t \|X(s)\|^2 ds + a_0 g_0 \int_0^t \|X(s)\|^2 ds\\ &\quad \leq \int_0^t |r(s)|\ \|X(s)\|\, ds.\end{aligned}$$

Hence, it is clear that

$$\begin{aligned}&\langle q(t)X'(t), X(t)\rangle - q_1 \int_0^t \|X'(s)\|^2 ds + \phi_0 \int_0^t \|X(s)\|^2 ds + a_0 g_0 \int_0^t \|X(s)\|^2 ds\\ &\quad \leq |\langle q(0)X'(0), X(0)\rangle| + \int_0^t |r(s)|\, \|X(s)\|\, ds.\end{aligned}$$

Let

$$K = \left|\left\langle q(0)X'(0), X(0)\right\rangle\right| + \left(\int_0^t |r(s)| \, \|X(s)\| \, ds\right).$$

Then, we obtain

$$\left\langle q(t)X'(t), X(t)\right\rangle - q_1 \int_0^t \left\|X'(s)\right\|^2 ds + (a_0 g_0 + \phi_0) \int_0^t \|X(s)\|^2 ds \le K.$$

By the Cauchy–Schwarz inequality for integrals, it follows that

$$\int_0^t |r(s)| \, \|X(s)\| \, ds \le \left(\int_0^t r^2(s) ds\right)^{1/2} \left(\int_0^t \|X(s)\|^2 ds\right)^{1/2}.$$

Let $n(t) = \left(\int_0^t \|X(s)\|^2 ds\right)^{1/2}$. When we divide both sides of the last inequality by $n(t)$, we find

$$n^{-1}(t)\{\left\langle q(t)X'(t), X(t)\right\rangle - q_1 \int_0^t \left\|X'(s)\right\|^2 ds\} + (a_0 g_0 + \phi_0)\left(\int_0^t \|X(s)\|^2 ds\right)^{1/2}$$
$$\le \frac{q(0)\left|\left\langle X'(0), X(0)\right\rangle\right|}{n(t)} + \left(\int_0^t r^2(s) ds\right)^{1/2}. \tag{11}$$

It can be seen that the right hand side of (11) is bounded and all terms on the left hand side of (11) are either bounded or positive. This case implies that the left hand side of (11) cannot be unbounded. Hence, we can conclude that $\|X(t)\|$ is square integrable. This completes the poof of Theorem 2.□

Example 2. Let

$$G(.) = \begin{bmatrix} \dfrac{1 + x_1^2 + x_2^2}{x_1} \\ \dfrac{1 + x_1^2 + x_2^2}{x_2} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad (x_1 \ne 0, x_2 \ne 0),$$

$$\Phi(.) = \begin{bmatrix} \dfrac{4 + x_1^2 + x_2^2 + x'^2_1 + x'^2_2}{x_1} \\ \dfrac{4 + x_1^2 + x_2^2 + x'^2_1 + x'^2_2}{x_2} \end{bmatrix},$$

It is clear that

$$\langle G(X), X\rangle \ge 2\|X\|^2, \quad g_0 = 2,$$

and

$$\left\langle \Phi(t, X, X'), X)\right\rangle \ge 4\|X\|^2, \quad \phi_0 = 4.$$

Hence, all the conditions of Theorem 2 hold. Therefore, all solutions of Eq. (3) are square integrable.

Remark. Napoles Valdes [10] proved his results, Theorems A and B, related to the boundedness and square integrability of solutions of Eq. (1) by the integral test. The results of Napoles Valdes [10] are new, the proofs presented in [10] simplify previous works since the Gronwall inequality is avoided which is the usual case. The proofs of our results are also done by the integral test. If we take $n = 1$ in Eq. (2) with $g(x) = x$, then our equation, Eq. (2), reduces to the equation discussed by Napoles Valdes [10], and in addition, our assumptions, the assumptions of Theorem 1, reduces to those of Napoles Valdes [10, Theorem 1]. For $n \ne 1$ in Eq. (2) when $g(x) = x$ in Eq. (1), Theorem 1 gives an n-dimensional generalization and extension of Napoles Valdes [10, Theorem 1], and here the Gronwall inequality is also avoided, which is the usual case.

Our second result, Theorem 2, also gives an n-dimensional dimensional generalization and extension of Napoles Valdes [10, Theorem 2] without using the Gronwall inequality. In this paper, since the Gronwall inequality is avoided here, which is the usual case, the results of this paper simplify previous works in the literature (see [1,5] and that in theirs references). Furthermore, our results extend and improve the results of [10, Theorem 1, Theorem 2] and those in the literature.

3. Conclusion

A certain class of non-linear differential systems of second order is considered. Two theorems are proved under suitable sufficient conditions: While the first theorem gives sufficient conditions guaranteeing that any solution of the considered equation as well as its derivative is bounded as $t \to \infty$, and the derivative of the considered solution is also square integrable, and the second theorem includes suitable sufficient conditions leads that all solutions of the considered equation are L^2 – solutions. The technique of proof used in the theorems involves the integral test and the Gronwall inequality is avoided in the proofs, which is not the usually expected. We give two examples to show the applicably of our results. The obtained results extend and improve some recent ones from scalar cases to the vectorial cases in the literature.

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